Tricritical point in heterogeneous k-core percolation

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k-core percolation is an extension of the concept of classical percolation and is particularly relevant to understand the resilience of complex networks under random damage. A new analytical formalism has been recently proposed to deal with heterogeneous k-cores, where each vertex is assigned a local threshold k_i. In this paper we identify a binary mixture of heterogeneous k-cores which exhibits a tricritical point. We investigate the new scaling scenario and calculate the relevant critical exponents, by analytical and computational methods, for Erdős-Rényi networks and 2d square lattices.

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Tricritical points (TCPs) are interesting critical phenomena in statistical physics, constituting a natural switch between 1$^{\text{st}}$ and 2$^{\text{nd}}$ order phase transitions. In other words, a TCP affords the possibility to smoothly control the order of the phase transition by tuning the appropriate parameter. In the language of critical phenomena, this is equivalent to identifying the position of a phase transition from the extrapolation of the order parameter- while usually impossible in 1$^{\text{st}}$ order transitions, it can be done in the case of continuous ones [1]. In the case of percolation models, there has recently been an attempt to identify a TCP in a model mixing elements of classical and explosive percolation in a lattice [2], although there is now evidence that the explosive percolation transition is continuous [3]. Other recent models which allow control of the order of the transition include explosive percolation on scale free networks [4], and dependency groups on interdependent networks [5]. In this paper we establish, for the first time, the presence of a TCP in a simple extension of classical percolation, namely heterogeneous k-core (HKC) percolation, which has the advantage of a sound analytical approach on random and complex networks [6, 7]. We show analytical evidence of a TCP in Erdős-Rényi graphs and numerically we find similar phase diagram topology in the square lattice. Finally, our model appears to be in the same universality class of a model which reproduces non-trivial signatures of liquid-glass transitions, including the higher-order glass singularity predicted by mode-coupling theory [8].

A k-core is defined as the maximal network subset which survives after a culling process which recursively removes all the vertices (and adjacent edges) with less than k neighbors. As a generalization of the concept of the giant component, the k-core gives a deeper insight into the structure and organization of complex networks. It has been thoroughly investigated on Bethe lattices [9], random graphs [6, 10] and, using a numerical approach, on various types of lattices [11]. The k-core percolation analysis has found several applications in varied areas of science including protein interaction networks [12], jamming [13], neural networks [14], granular gases [15] and evolution [16]. Important insights into the resilience of networks under damage [17] and spreading of influence in social networks [18] are gleaned from an understanding of the k-core structure of the network. As in Ref. [6, 19], we can study k-core percolation on networks after randomly removing a fraction $1-p$ of vertices. We use the treelike properties of the configuration model [20], in which the number of loops vanishes as $N \to \infty$, which guarantees that if a k-core exists, it must be infinite, at least if $k \geqslant 2$ [6, 9]. In the HKC extension [7] each vertex has its own threshold and the culling process is based on local, vertex-dependent rules. Although Baxter *et al.* developed results for an arbitrary distribution of vertex thresholds, they study binary mixtures of vertices of types a and b, with thresholds $k_a = 1$, $k_b \geqslant 3$. The first heterogeneous models of this kind were investigated by Branco [21] on a Bethe lattice, whereas the related problem of bootstrap percolation (BP) has been much studied on regular lattices [11, 22]. Here we focus on the case $\mathbf{k} \equiv (k_a, k_b) = (2, 3)$.

We start with a binary mixture (k_a,k_b), where vertices have been randomly assigned two thresholds k_a and k_b (say $k_a < k_b$) with probability r and $1-r$, respectively. Finite clusters are a possibility when $k_a = 1$ and so we must make a distinction between M_{ab}, the probability that a randomly chosen vertex belongs to the HKC, and $\mathcal{S}_{ab}$, the probability that a randomly chosen vertex belongs to the *giant* component of the HKC. We will show that in the case $\mathbf{k} = (2, 3)$ these two quantities are coincident, but there are relevant examples where they are not [7].

In the original k-core formalism, given the end of an edge, a $(k-1)$-ary subtree is defined as the tree where, as we traverse it, each vertex has at least $k-1$ outgoing edges, apart from the one we came in. Instead, considering a HKC, every vertex i may have a different threshold k_i. The $(k_i - 1)$-ary subtree, then, is the tree in which, as we traverse it, each encountered vertex has at least k_i-1 child edges. We define Z as the probability that a randomly chosen vertex is the root of a $(k_i - 1)$-ary subtree. Taking advantage of the local treelike nature of the configuration model, Z is related to M_{ab} as [7]:

$$M_{ab}(p) = \bar{M}_a(p) + \bar{M}_b(p) = pr\sum_{q=k_a}^{\infty} P(q)\Phi_q^{k_a}(Z,Z) + \\ +p(1-r)\sum_{q=k_b}^{\infty} P(q)\Phi_q^{k_b}(Z,Z) \quad (1)$$

where $\bar{M}_{a(b)}(p)$ is the fraction of nodes of type $a(b)$ in the HKC, respectively, $P(q)$ is the degree distribution and we have used the convenient auxiliary function:

$$\Phi_q^k(X,Z) = \sum_{l=k}^{q}\binom{q}{l}(1-Z)^{q-l}\sum_{m=1}^{l}\binom{l}{m}X^m(Z-X)^{l-m}.$$

The quantity $\Phi_q^{k_{a(b)}}(Z,Z)$ in (1) represents the probability that a vertex of type $a(b)$ of degree q has at least $k_a(k_b)$ edges which are roots of a (k_i-1)-ary subtree. This quantity is summed over all possible degrees, taking account of the relevant vertex type fraction. The self-consistent equation for Z is:

$$Z = pr\sum_{q=k_a}^{\infty}\frac{qP(q)}{\langle q\rangle}\Phi_{q-1}^{k_a-1}(Z,Z) + \\ +p(1-r)\sum_{q=k_b}^{\infty}\frac{qP(q)}{\langle q\rangle}\Phi_{q-1}^{k_b-1}(Z,Z) \quad (2)$$

We now consider the probability X, that a randomly chosen edge leads to a vertex which is the root of an *infinite* (k_i-1)-ary subtree. In the case of a binary mixture, X is written as [7]

$$X = pr\sum_{q=k_a}^{\infty}\frac{qP(q)}{\langle q\rangle}\Phi_{q-1}^{k_a-1}(X,Z) + \\ +p(1-r)\sum_{q=k_b}^{\infty}\frac{qP(q)}{\langle q\rangle}\Phi_{q-1}^{k_b-1}(X,Z), \quad (3)$$

The fraction of vertices in the giant HKC $\mathcal{S}_{ab}$, then, is given by $\mathcal{S}_{ab}(p) = \bar{\mathcal{S}}_a(p) + \bar{\mathcal{S}}_b(p)$, where the fraction of nodes of type a is $\bar{\mathcal{S}}_a(p) = pr\sum_{q=k_a}^{\infty} P(q)\Phi_q^{k_a}(X,Z)$ and an analogous expression holds for $\bar{\mathcal{S}}_b(p)$.

For $k_a = 1, k_b \geqslant 3$ mixtures on the Bethe lattice, the phase diagram shows a critical line which meets a first order line at a critical end point and a critical point at the end of a two-phase coexistence between a low and a high density phase [7]. The two lines do not match at a TCP, because the 1-nodes are so robust that a 1-rich phase is stable to damage at intermediate compositions even when the k_b-rich phase has collapsed. Let us consider now the case $\mathbf{k} = (2,3)$, with a degree distribution such that $\sum_q q^2P(q) < \infty$. We can rewrite Z (Eq. 2) as $pf(Z) = 1$ where

$$f(Z) = r\frac{2P(2)}{\langle q\rangle} + \sum_{q\geq 3}\frac{qP(q)}{\langle q\rangle}\times \\ \times\left[\frac{1-(1-Z)^{q-1}}{Z} - (1-r)(q-1)(1-Z)^{q-2}\right] \quad (4)$$

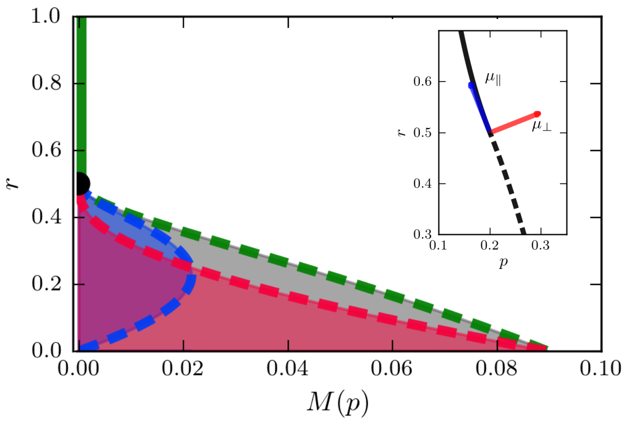

FIG. 1. Phase diagram of the $\mathbf{k} = (2,3)$ mixture, showing the total mass of the percolating HKC cluster at different compositions r, for ER networks with $z_1 = 10$. The TCP at $r = 1/2$ separates a line of 1$^{\text{st}}$ order transitions (dashed) from the 2$^{\text{nd}}$ order line (solid). The masses of the 2-rich-core (blue) and the 3-rich-core (red) in the giant HKC are also shown. The inset shows the phase diagram in the (r,p) space.

and similarly rewriting Eq. 3 as $h(X,Z) = 1/p$ with

$$h(X,Z) = r\frac{2P(2)}{\langle q\rangle} + \sum_{q\geq 3}\frac{qP(q)}{\langle q\rangle}\times \\ \times\left[\frac{1-(1-X)^{q-1}}{X} - (1-r)(q-1)(1-Z)^{q-2}\right]. \quad (5)$$

These two equations differ only in the first (fractionary) part of the sum. The X-dependent (positive) general term of the series is monotonically decreasing for any $0 < X \leqslant 1$, meaning that Eq. (3) has only one non-zero solution when Eq. (2) has a non-zero solution and therefore $X = Z$ for the $\mathbf{k} = (2,3)$ mixture (and $S_{23} = M_{23}$). We expect this property to be true for any mixture with nodes of type $k \geqslant 2$.

We now explicitly show that the $\mathbf{k} = (2,3)$ mixture presents a TCP for an Erdős-Rényi (ER) degree distribution with mean degree z_1 $P(q) = z_1^q\exp(-z_1)/q!$. Using the condition $X = Z$, the equation $pf(Z) = 1$ fully solves the problem of finding the onset of the giant HKC, and the function $f(Z)$ becomes $f(Z) = \{1 - e^{-z_1 Z}[1 + (1-r)z_1 Z]\}/Z$. It is now clear that $f'(Z) < 0$ for every $r > \frac{1}{2}$, implying that the only solution is the trivial one $Z = 0$, with a de-percolating 2$^{\text{nd}}$ order phase transition occurring at the critical occupancy probability $p_c = 1/rz_1$. For $r < \frac{1}{2}$, $f(Z)$ has a maximum at $0 < Z_M < 1$. This implies the presence of a 1$^{\text{st}}$ order transition and a coexistence between a HKC phase of strength $M_{23}(Z_M)$, given by (1), and the non percolating phase at $Z = 0$. The expansion of $f(Z)$ for $r \geqslant \frac{1}{2}$, $Z(p) \to 0$, as $p \to p_c^+$, yields $f(Z) = rz_1 + (1/2 - r)z_1^2 Z + O\left(Z^2\right)$ showing that the maximum of $f(Z)$ continuously matches the $Z = 0$ line exactly at $r_t = \frac{1}{2}$, where a TCP is present.

We show the computed phase diagram of the $\mathbf{k} = (2, 3)$ mixture in Fig. 1.

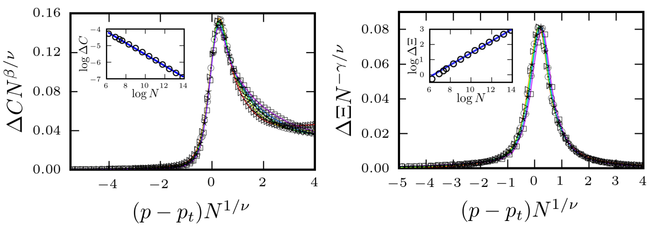

FIG. 2. Re-scaling of the corona mass C_{23} and the mean corona cluster size Ξ_{23} at the TCP on ER networks. The data range in size from $N = 2^9$ to $N = 2^{18}$ via successive doublings. We find the exponent ratios $\beta/\nu = 0.34(5)$ and $\gamma/\nu = 0.39(4)$ from the scaling of C_{23} and Ξ_{23} at p_t (insets) and show the data collapse achieved with those exponents (main panels $N = 2^{14} \ldots 2^{18}$; $\triangleright, \odot, \star, \oplus, \square$, respectively).

We now calculate the critical exponents for this mixture, in particular at the TCP at $r_t = \frac{1}{2}$, $p_t = \frac{2}{z_1}$. The expansion of the order parameter $M_{23}(p)$ for $p \to p_c^+$ at $r \geqslant \frac{1}{2}$ and $p \to p^{*+}$ (the border of the coexistence region) at $r < \frac{1}{2}$ yields three different values for the exponent β:

$$\beta = \begin{cases} 2 & 1/2 \leqslant r < 1 \\ 1 & r = 1/2 \\ 1/2 & 0 \leqslant r < 1/2 \end{cases} \tag{6}$$

The exponent β takes a unique value at the TCP, and agrees with the values found by Branco on the Bethe lattice [21]. However, in this work the presence of finite size cores had not been properly handled and it was erroneously assumed that the phase diagrams of the $\mathbf{k} = (1, 3)$ and the $\mathbf{k} = (2, 3)$ mixture had the same topology. The exponent $\beta = 1/2$ for $r < 1/2$ corresponds to the usual hybrid phase transition seen in k-core percolation, a discontinuous transition which combines with critical fluctuations (only on the percolating side) as usually found in 2$^{\text{nd}}$ order transitions. To our knowledge, the $\mathbf{k} = (2, 3)$ mixture is the first model displaying a TCP adjacent to a hybrid phase transition.

It has been shown that subsets of the HKC called corona clusters have the same critical properties of the HKC [13, 19]. The corona vertices have exactly k_i neighbours in the HKC, and form finite clusters whose mean size diverges when approaching the threshold from above. The corona clusters provide a more convenient order parameter for numerical study of the model on random networks, in contrast to the HKC where only one (infinite) cluster survives. Using the configuration model with ER degree distribution we simulated the $\mathbf{k} = (2, 3)$ mixture for various sizes. The typical ansatz of finite size scaling for a continuous transition is that any quantity Y scaling as $Y \sim (p - p_c)^{-\chi}$ should have the form $Y = N^{\chi/\nu} F\left[(p - p_c)N^{1/\nu}\right]$, where ν is the correlation length exponent and F is some scaling function. Given the universal nature of F we expect to see data collapse in a plot of $Y N^{-\chi/\nu}$ against $(p - p_c)N^{1/\nu}$. Computing the mass of the heterogeneous corona $C_{23}(k)$ at the TCP for various sizes we find $\beta/\nu = 0.34(5)$ (Fig. 2). Similarly for the mean corona cluster size Ξ_{23}, we find $\gamma/\nu = 0.39(4)$. We determine the exponent $\nu = 2.86(9)$ by the scaling of the effective percolation threshold with size $p_{ave} - p_c \sim N^{-1/\nu}$, where we have located p_{ave} from the peak of the susceptibility of the corona mass $\Delta C_{23} = (\langle C_{23} \rangle^2 - \langle C_{23} \rangle^2)^{1/2}$. We find good data collapse with these exponents in the scaling window at the TCP (Fig. 2), and fit the exponents $\beta = 0.9(90)$ and $\gamma = 1.13(1)$, the former being close to the value calculated analytically. The behavior of the strength of the HKC along the edges of the coexistence region near the TCP for $r \to \frac{1}{2}^+$ allows us to calculate analytically the subsidiary tricritical exponent β_u defined by $M^*(r) \sim \left(\frac{1}{2} - r\right)^{\beta_u}$ [23]. For $\mathbf{k} = (2, 3)$ we find $\beta_u = 2$.

The tricritical crossover exponent φ_t describes the change of the critical line as the TCP is approached [24]. Thus, we write the critical line in terms of two scaling fields $\mu_\perp$ and $\mu_\parallel$, perpendicular and tangent to the critical line, respectively. Given the simplicity of the model, this calculation can also be done analytically for ER networks. The rotation defining the critical fields is

$$\begin{pmatrix} \mu_\perp \\ \mu_\parallel \end{pmatrix} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} p - p_t \\ r - r_t \end{pmatrix} \tag{7}$$

with $\tan\vartheta = 4/z_1$. Close to the TCP, the critical line has a behavior $\mu_\parallel \sim \mu_\perp^{1/2}$, with a crossover exponent $\varphi_t = 2$ (Fig. 1). We expect that the above critical behavior (as well as the values of the critical exponents) is reproduced by all degree distributions with finite second moment.

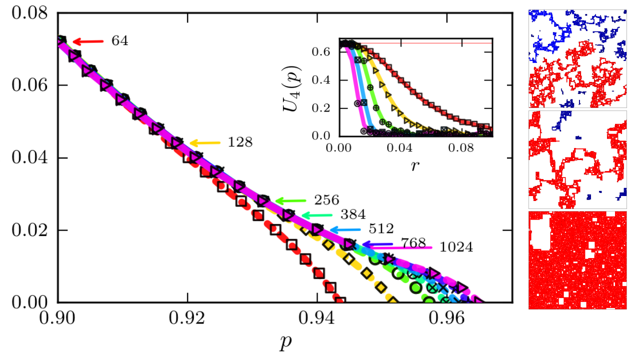

FIG. 3. Phase diagram for the $\mathbf{k} = (2, 3)$ lattice model showing the threshold density p against composition r for several sizes $L = 64(\square)$, $128(\diamond)$, $256(\circ)$, $384(\otimes)$, $512(\times)$, $768(\star)$, $1024(\triangleright)$. The arrows indicate the location of the TCP for each L. The inset shows the Binder cumulant $U_4(p_c)$ for various sizes, indicating the narrowing of the tricritical region for increasing size. On the right are sample configurations at the threshold density in the continuous transition region (top), at the TCP (centre) and in the discontinuous region (bottom); in each case the critical HKC cluster is red.

We simulated the $\mathbf{k} = (2,3)$ model on a 2d square lattice and located the TCP at small concentrations of $k = 2$ vertex types (Fig. 3). On the lattice, the analogous bootstrap percolation (BP) model has been much studied and it is known [25] that for $k \geqslant d+1$ a discontinuous transition occurs only at $p = 1$. For $k \leqslant d$ the transition is continuous, although the critical exponents have values which in general depend on k. On the continuous side of the TCP we can expect the usual scaling of the threshold density $p_{ave} - p_c \sim L^{-1/\nu}$, whereas for $r = 0$ we might expect the scaling form found in BP $p_{ave} - p_c \sim 1/\log L$ [26], although numerical simulations have struggled to confirm this scaling in several cases [11]. As shown in Fig. 3 the TCP moves toward $r = 0$ with increasing size (determination of the precise scaling with L requires far larger sizes and is the subject of further work). In fact, there is a finite window of r over which the transition slowly changes from 1^{st} to 2^{nd} order, and this window becomes sharper with increasing system size. We quantify this with the Binder cumulant $U_4(p_c) = 1 - \langle M \rangle^4 / 3 \langle M^2 \rangle^2$ which has the value $U_4 = \frac{2}{3}$ on the 1^{st} order side and 0 on the 2^{nd} order side (inset of Fig. 3). Data collapse near the TCP does not work due to the presence of different scaling regimes. We determined the critical exponents at the TCP for the largest size simulated ($L = 1024$) and found the exponents $\beta = 0.31(5)$, $\gamma = 2.51(3)$ and $\nu = 1.39(9)$. Exponents γ and ν at the TCP are very close to their values for ordinary percolation on a 2d lattice (and a little smaller than the ones of explosive percolation [2]). Exponent β, instead, is significantly larger. The fractal dimension of the tricritical HKC clusters is $D = 2 - \beta/\nu = 1.77(8)$, somewhat smaller than ordinary percolation ($D = 1.879$), reflecting the presence of large, jagged voids in the $\mathbf{k} = (2,3)$ mixture at the TCP. The unusual finite-size effects in this model are reflected in a violation of the hyperscaling relation.

In contrast with the $\mathbf{k} = (2,3)$ case, the phase diagrams for $k_a = 1$, $k_b \geqslant 3$ mixtures [7] do not present a TCP. Moreover, the analytical properties of $f(Z)$ and $h(X,Z)$ indicate that TCPs are also absent in mixtures of type $k_a = 2, k_b > 3$. Though far from ubiquitous, a TCP is indeed present in the $\mathbf{k} = (2,3)$ mixture, not only on the Bethe lattice but also in ER graphs and regular square lattices. This case appears to be peculiar because the resiliences of the 3-rich-phase and the 2-rich-phase are sufficiently close that the two phases collapse at the same damage fraction, leading to a complete failure of the HKC, either through a 1^{st} or a 2^{nd} order transition. This phenomenon may occur in cases when a mixing of a continuously and a discontinuously failing phase is not too heterogeneous ($k_b = k_a + 1$). It is intriguing to note that the case $\mathbf{k} = (2,3)$ almost exactly maps onto a model of glasses, recently studied on the Bethe lattice [8], which appears to be in the same universality class.

In conclusion, we have presented a new model of HKC percolation which supports a smooth interpolation between classical percolation and a 1^{st} order phase transition through a TCP. We are able to identify a new tricritical scaling scenario and calculate, both by analytical and numerical methods, critical exponents which are different from the ones of known percolation transitions. We prove the presence of this critical phenomenon in ER graphs, and we also get strong numerical evidence in the square lattice. The capacity to govern the order of phase transitions in randomly damaged networks may constitute a step towards a more effective infrastructure for network protection.

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